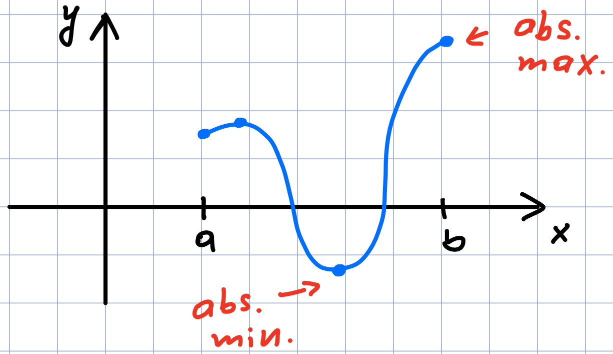
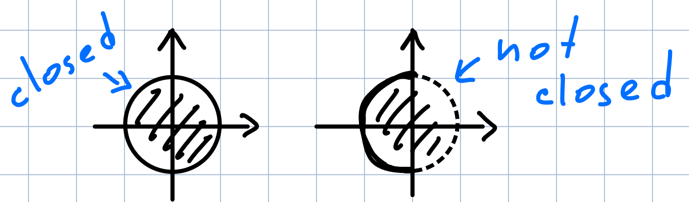
Last time: Extrema on bounded regions

Recall: $f(x)$ continuous on $[a,b]$, has an absolute max. and an abs. min. on $[a,b]$.



- We find them by evaluating f on critical points AND on endpoints.

$\mathcal{D}$ - closed set in $\mathbb{R}^2$ (i.e. contains boundary pts); bounded set (contained in some disk)



f, continuous on a closed, bounded $\mathcal{D} \subset \mathbb{R}^2$, attains an abs. max. value $f(x_1, y_1)$ and an abs. min. value $f(x_2, y_2)$ at some points in $\mathcal{D}$.

To find abs. max/min values of f on a closed, bounded $\mathcal{D}$:

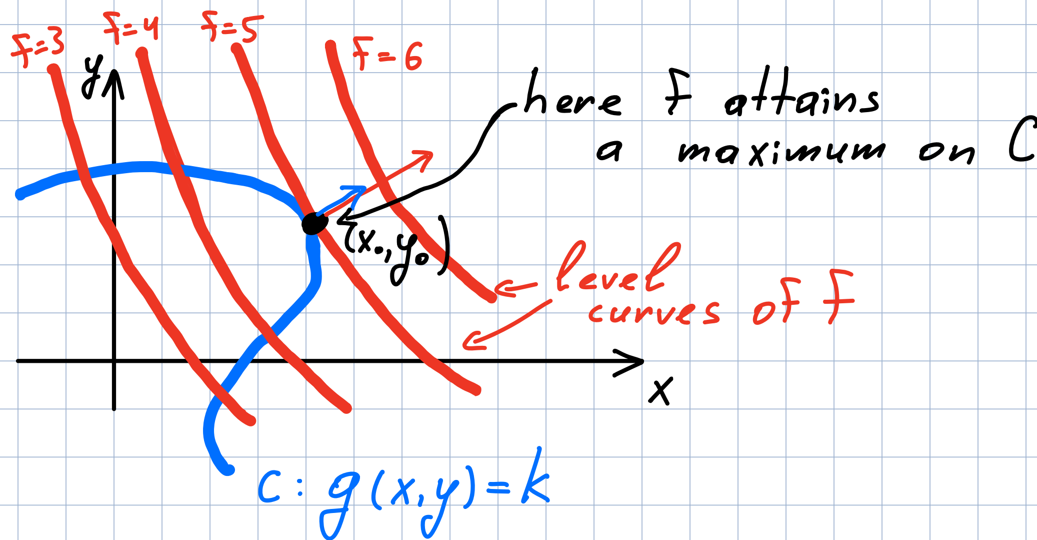
1. Find values of f at critical points in $\mathcal{D}$;
2. Find extreme values on the boundary of $\mathcal{D}$;
3. largest of 1, 2 – abs. max.
   smallest of 1, 2 – abs. min.

Lagrange multipliers

Problem:

Find extreme values of $f(x,y)$ subject to constraint $g(x,y)=k$

[or: $f(x,y,z)$ subject to $g(x,y,z)=k$]



level curves $g(x,y)=k$ and $f(x,y)=6$ touch at (x_0,y_0)

$\Rightarrow$ they have the same normal line.

$\Rightarrow$ $\nabla f(x_0,y_0)$ is parallel to $\nabla g(x_0,y_0)$

i.e. $\nabla f(x_0,y_0) = \underbrace{\lambda}_{} \nabla g(x_0,y_0)$

a number, "Lagrange multiplier"

Rmk: Similarly for 3 variables: $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$

level surfaces of f and g are touching

Method of Lagrange multipliers

To find maximum & minimum values of $f(x,y,z)$

subject to constrain $g(x,y,z)=k$:

ⓐ find all values of x, y, z, λ such that

- $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$
- $g(x,y,z) = k$

$$\boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \end{aligned}}$$

ⓑ evaluate f at all points found in (a)

maximum = largest value of f

minimum = smallest

Rmk: Similarly for 2 variables.

Ex: Find extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2+y^2=1$.

Sol: $g(x,y) = x^2+y^2 = 1$ - constraint

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{cases} \quad \text{or} \quad \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 1 \end{cases}$$

$$\begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2+y^2=1 \end{cases} \Rightarrow \quad x=0 \quad \text{or} \quad \lambda = 1$$

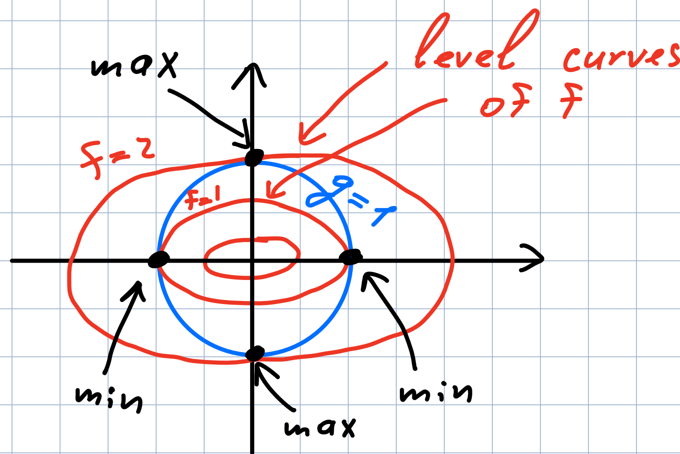
- if $x=0$, then $x^2+y^2=1 \Rightarrow y=\pm 1$,
  and $4y=\lambda 2y \Rightarrow \lambda = 2$
- if $\lambda=1$, then $4y=\lambda 2y \Rightarrow y=0$,
  and $x^2+y^2=1 \Rightarrow x=\pm 1$

We obtain 4 points:

$(0,1)$	$(0,-1)$	$(1,0)$	$(-1,0)$
$f(0,1)=2$	$f(0,-1)=2$	$f(1,0)=1$	$f(-1,0)=1$

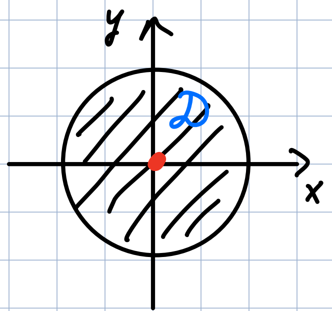
So, maximum: 2

minimum: 1



Ex: Find extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2+y^2 \le 1$

Sol:



1) Critical pts:

$$\left.\begin{array}{l} f_x = 2x = 0 \\ f_y = 4y = 0 \end{array}\right\} \Rightarrow \quad (x,y) = (0,0) \text{ crit. pt.} \qquad f(0,0) = \underline{0}$$

2) Extreme values of f on the boundary:

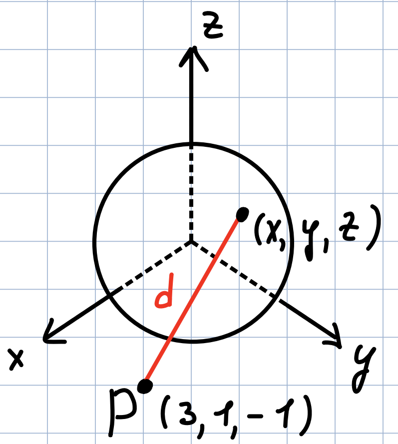
(From previous example)

$$f(\pm 1, 0) = \underline{1}, \quad f(0, \pm 1) = \underline{2}$$

So, $\boxed{\begin{array}{l} \text{maximum on } \mathcal{D}: \ f(0, \pm 1) = 2 \\ \text{minimum on } \mathcal{D}: \ f(0,0) = 0 \end{array}}$

Ex: Find the points on the sphere $x^2+y^2+z^2=4$ closest to / farthest from the point $P(3,1,-1)$

Sol:



1) $d=\sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$ – find extreme values
– where they are attained?

2) It is easier to work with d^2 than with d

$$d^2=(x-3)^2+(y-1)^2+(z+1)^2 = f(x,y,z)$$

WANTED: extreme values of f subject to

$$g(x,y,z)=x^2+y^2+z^2=4.$$

3) $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z)=4 \end{cases}$

$$\begin{cases} 2(x-3) = \lambda\, 2x \\ 2(y-1)=\lambda\, 2y \\ 2(z+1)=\lambda\, 2z \\ x^2+y^2+z^2=4 \end{cases} \iff \begin{cases} x(1-\lambda)=3 \\ y(1-\lambda)=1 \\ z(1-\lambda)=-1 \\ x^2+y^2+z^2=4 \end{cases} \quad \begin{cases} x=\frac{3}{1-\lambda} \\ y=\frac{1}{1-\lambda} \\ z=-\frac{1}{1-\lambda} \\ \frac{1}{(1-\lambda)^2}\left(3^2+1^2+(-1)^2\right)=4 \end{cases}$$

$$\Rightarrow (1-\lambda)^2=\frac{11}{4} \Rightarrow 1-\lambda=\pm\frac{\sqrt{11}}{2} \Rightarrow \lambda = 1\mp\frac{\sqrt{11}}{2}$$

4) $\lambda = 1 + \frac{\sqrt{11}}{2} \Rightarrow (x, y, z) = \overbrace{\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}^{\text{farthest point from P on the sphere}}$

$$d^2 = (\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2 = \left(\frac{3\lambda}{1-\lambda}\right)^2 + \left(\frac{\lambda}{1-\lambda}\right)^2 + \left(\frac{-\lambda}{1-\lambda}\right)^2 = \frac{11\,\lambda^2}{(1-\lambda)^2} = \frac{11 \cdot \left(1 + \frac{\sqrt{11}}{2}\right)^2}{11/4} = (2 + \sqrt{11})^2$$

$\lambda = 1 - \frac{\sqrt{11}}{2} \Rightarrow (x, y, z) = \underbrace{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)}_{\text{closest pt. to P on the sphere}}$

$$d^2 = (\sqrt{11} - 2)^2$$